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ON A PHASE AUTOMATIC FREQUENCY CONTROL EQUATION WITH A LAG AND A RECTANGULAR PHASE DETECTOR CHARACTERISTIC

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A second-order piecewise-linear dynamic system with jumps in the representative point on the juncture lines is investigated on a cylindrical phase space.

We consider the equation

$$\frac{d^2\varphi}{dt^2}+h\,[1-bF'(\varphi)]\frac{d\varphi}{dt}+F(\varphi)=\gamma \qquad \left(\begin{matrix} b>0,\ h>0 \\ 0\leqslant\gamma<1\end{matrix}\right)$$

$$F(\varphi+2k\pi)\equiv F(\varphi)\quad (k=0,\pm1,\ldots),\quad F(\varphi)=\begin{cases}-1 & \text{for} \quad -\pi<\varphi<0\\ 1 & \text{for} \quad 0<\varphi<\pi\end{cases}$$

This equation describes the dynamics of a phase automatic frequency control (afc) system with an integrating filter [1, 2] and a rectangular phase detector characteristic [3] with an approximate accounting for the lag [1]. It has no meaning for values of φ at which $F(\varphi)$ suffers discontinuities. By introducing new variables and notation

$$t^\circ=ht,\quad y=\frac{1}{h}\frac{d\varphi}{dt},\quad \alpha=\frac{h^2\pi}{(1+\gamma)},\quad \beta=\frac{h^2\pi}{(1-\gamma)}\geqslant\alpha$$

we replace the equation in the strips $-\pi<\varphi<0$ and $0<\varphi<\pi$ by the systems

$$\varphi^{\cdot}=y,\qquad y^{\cdot}=\alpha^{-1}\pi-y\quad(-\pi<\varphi<0) \tag{1}$$

$$\varphi^{\cdot}=y,\qquad y^{\cdot}=-\beta^{-1}\pi-y\quad(0<\varphi<\pi) \tag{2}$$

Here the dots denote differentiation with respect to t° ; a cylinder serves as the phase space of the system. Systems (1) and (2) permit us to trace the motion of the representative point upto the instant when it hits onto one of the straight lines $\varphi=0$ or $\varphi=\pi$. The subsequent motion of the representative point requires an extension of the definition. We should indicate how much time it spends on the straight line, how it moves along it, at which point it leaves, and which of systems (1) or (2) describes its subsequent motion. We make use of the extended definition given in [4]. (When applying the formula (*) in [4] it is necessary to take into account the scales of t and y).

*) In [4] (English Version), page 756, line four from the top the erroneous equation $r=2b,\ h>0$, as given in the Russian Original Edition, should read $r=2bh>0$.

The scheme for the extended motions on the straight line $\varphi = \pi$ is shown in Fig. 1, a. Here, for the sake of vizualization the mutually overlapping trajectories on the φ-axis have been moved apart. The representative point, having hit upon the point (π, y) where $y \geqslant 2b$, makes a jump of size $2b$ downward along the straight line $\varphi = \pi$, after which the motion continues for $\varphi = \pi$ in accordance with system (1). If $b(1-\gamma) < y < 2b$, the representative point skips into the point $(\pi, 0)$ and is located at it for the time

$$bh^2 \ln\{b(1+\gamma)/[y - b(1-\gamma)]\}$$

after which the motion continues for $\varphi > \pi$ in accordance with system (1). If $y = b(1-\gamma)$, the representative point skips into the point $(\pi, 0)$ and remains there for an unboundedly long time. If $0 \leqslant y < b(1-\gamma)$, the representative point skips into the point $(\pi, 0)$ and is located at it for the time

$$bh^2 \ln b(1-\gamma)/b(1-\gamma) - y$$

after which the motion continues for $\varphi < \pi$ in accordance with system (2). The behavior of the representative point moving in the lower half-cylinder and hitting onto the straight line $\varphi = \pi$ is analogous (we need only replace y by $-y$ and γ by $-\gamma$).

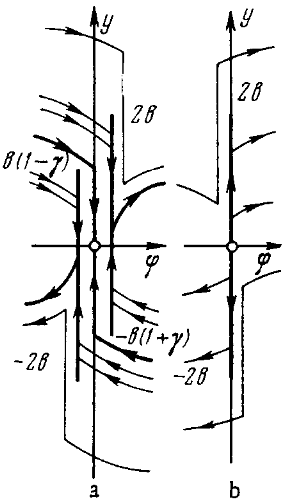

Fig. 1.

The point $(\pi, 0)$ is an equilibrium point analogous to a saddle, while the trajectories passing through the points $[\pi, b(1-\gamma)]$ and $[\pi, -b(1+\gamma)]$ are its separatrices. The role of the other two separatrices is played by the trajectories issuing from the point $(\pi, 0)$.

The scheme for the extended motions along the straight line $\varphi = 0$ is shown in Fig. 1, b. The representative point, having hit onto the point $(0, y)$, where $y > 0$, makes a jump of size $2b$ upward along the straight line $\varphi = 0$, after which motion continues for $\varphi > 0$ in accordance with system (2). If $y < 0$, the representative point makes a jump of size $2b$ downward along the straight line $\varphi = 0$, after which motion continues for $\varphi < 0$ in accordance with system (1). The point $(0, 0)$ is an equilibrium state analogous to an unstable node. From it issue trajectories passing through the point $(0, y)$, where $0 < y \leqslant 2b$ or $-2b \leqslant y < 0$. (The representative traverses the vertical part of such a trajectory from the point $(0, 0)$ to the point $(0, y)$ by a jump). When the system has been completely defined on the straight lines $\varphi = 0$ and $\varphi = \pi$, we can trace any particular solution of it on any interval of time and investigate it qualitatively. Not doing this here (it is similar to the investigation in [5]), we limit ourselves to a presentation of the results.

Figure 2 shows the schematic partitioning of the space of parameters α, β, $k = bh^2$ of the system being investigated into regions of different qualitative structure of the phase trajectories (in the section $\beta = \text{const}$) of the plane). This partitioning is effected by the heavy lines in Fig. 2. The corresponding structures are presented in Fig. 3. For simplicity on Fig. 3 we have taken a nonuniform scale on the φ-axis, and both the equilibrium states are shown on the front half of the cylinder; on a uniform scale one of them would be on the back half. In region 1 the system has a stable limit cycle surrounding the equilibrium state $(0, 0)$ (shrinking to this equilibrium state uniformly as

$k \to 0$). The thin lines partition region 1 into subregions distinguishing the nature of the limit cycle. In subregion 1, a the cycle has two vertical segments on the straight line $\varphi = 0$, which the representative point traverses by a jump. In subregion 1, b the cycle has a further vertical segment on the upper part of the straight line $\varphi = \pi$. The representative traverses it also by a jump, but before motion continues on the lower part of the cycle, it dwells for some time at the point $(\pi, 0)$. In subregion 1, c the same kind of vertical segment of the cycle occurs also on the lower part of the straight line $\varphi = -\pi$.

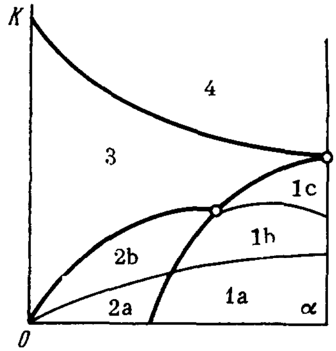

Fig. 2.

When passing from region 1 to region 2, a stable limit cycle girding the upper half-cylinder is born from the loop formed by the separatrix of the equilibrium state $(\pi, 0)$. Figure 4, a shows (on a developed cylinder) the instant of birth of the cycle, and Fig. 4, b shows the cycle. On the straight lines $\varphi = 0$ and $\varphi = \pi$ it has vertical segments which the representative point traverses by a jump. After the jump along the straight line $\varphi = \pi$ the representative point makes a stop at the point $(\pi, 0)$ before motion continues. Thus, in region 2 we have a stable limit cycle girding the upper half-cylinder and a stable cycle surrounding the equilibrium state (0, 0). The latter cycle can be of two types. In subregion 2, a it is just as in 1, a; in subregion 2, b it is just as in 1, b.

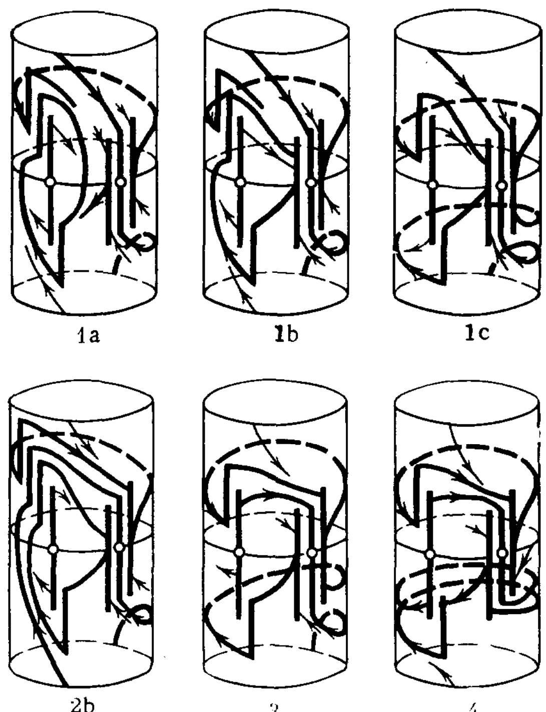

Fig. 3.

When passing from region 2 to region 3 this cycle becomes a loop formed by the separatrix of the equilibrium state $(\pi, 0)$. In region 3 the system has only one cycle, girding the upper half-cylinder. When passing from region 1 into region 3 through the boundary 1, c - 3 the cycle surrounding the equilibrium state (0, 0) becomes a loop formed by the separatrix of the equilibrium state $(\pi, 0)$. Simultaneously, a stable limit cycle girding the upper half-cylinder is born from the part of this loop lying in the upper half-cylinder.

When passing from region 3 into region 4 a stable limit cycle girding the lower half-cylinder is born from the loop formed by the separatrix of the equilibrium state $(\pi, 0)$. Thus, in region 4 the system has two stable limit cycles: one in the upper and another

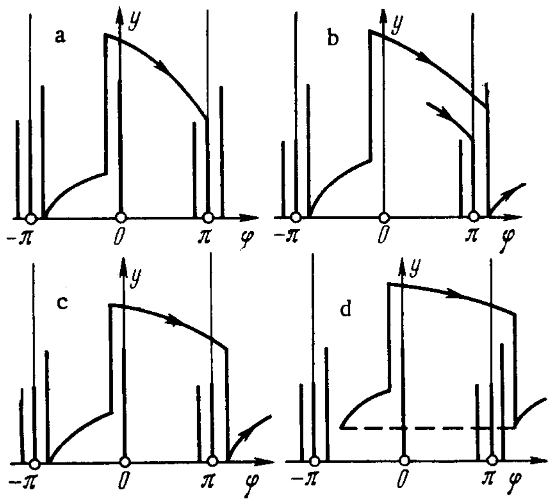

Fig. 4.

in the lower half-cylinder. The region of existence of a limit cycle girding the upper half-cylinder (the aggregate of regions 2, 3, 4) can be divided into two parts distinguishing the nature of the cycle. In the part abutting the boundary $1-2$, $1-3$ it has the nature described above and is shown in Fig, 4, b. When passing to the other part (the transition instant is shown in Fig. 4, c) the cycle "separates" from the φ-axis (see Fig. 4, d). When moving along this cycle the representative point has no stops. The region of existence of a cycle girding the lower half-cylinder can be similarly divided into two parts.

In conclusion we cite the equations, of interest in practice, of the surfaces $1-2$, $1-3$ corresponding to the birth of a limit cycle in the upper half-cylinder

$$\alpha = e^{-\tau} + \tau - 1, \ \beta = (k+1)(e^{\theta} - 1) - \theta, \ \beta\tau - \alpha\theta = 2\alpha\beta - k\beta$$

Here τ and θ are subject to elimination.

Note. This paper is closely related to the author's papers [4, 5] and to those of other authors [6 - 8] in which the given equation is considered under a number of assumptions on the sign of the coefficient b of the form of the discontinuous function $F(\varphi)$. By comparing the results in [4, 5] with those herein on the one hand, and with the results in [6 - 8] on the other, we can see the following: 1) the methods applied in [6 - 8] do not allow us to establish completely the nature of all the solutions of the equation, in particular, in some cases it is not possible to compute the transient responses ending at the equilibrium state; 2) the methods employed in [7, 8] can lead to improper conclusions in some cases. Thus, if we apply them to the case investigated here, the cycle shown in Fig. 4, b is lost, and as the condition for the birth of the cycle we have to accept the condition for the conversion of a cycle of type 4, b into a cycle of type 4, d.

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